This exam consists of **two problems**, worth a total of 40 points.

Problem 1 has five parts, and is worth 25 points.
Problem 2 has three parts, and is worth 15 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- The Q-function is defined as
$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \, dt,$$
for $x \geq 0$.
- The following bound on the Q-function is valid:
$$Q(x) \quad \leq \quad \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \geq 0$$

1) Consider a bandpass transmission system that uses two signaling waveforms, given by

$$\begin{aligned} s_1(t) &= A_c k \sin(2\pi f_c t) + A_c\sqrt{1-k^2}\cos(2\pi f_c t) \quad \text{and} \\ s_2(t) &= A_c k \sin(2\pi f_c t) - A_c\sqrt{1-k^2}\cos(2\pi f_c t), \end{aligned}$$

for $0 \le t \le T_b$, where $A_c > 0$ and $k \in [0,1)$ are parameters, and $f_c = n/T_b$ for some fixed integer n. (The first 'sin' term may be interpreted as a "carrier" component.)

a) [3 points] Find an orthonormal basis for these waveforms, and sketch a signal space diagram for this transmission scheme.

b) [4 points] Suppose that the communication system is coherent, and the received waveform is corrupted by additive white Gaussian noise with zero mean and power spectral density $N_0/2$. Find an expression for the average probability of (symbol) error, assuming that the signals are equally likely. Express your answer in terms of the Q-function, parameterized by k, N_0, and the energy per bit

$$E_b = \frac{1}{2}A_c^2 T_b.$$

c) [5 points] In the same setting as problem (b) above, suppose that k is chosen so that 10% of the transmitted signal power is allocated to the carrier component. Find a *sufficient* condition on the quantity E_b/N_0 to ensure that the probability of error does not exceed 10^{-4}. Use the upper bound on the Q-function provided on the previous page.

d) [6 points] Suppose that you want to modify the binary signaling scheme introduced above by adding one *additional* signaling waveform (and keeping the other two fixed), and that the new collection of waveforms will be equally likely. Specify this new waveform, under the conditions that the corresponding signal space representations for the new collection of 3 waveforms be equally spaced, and that the resulting constellation has the smallest average transmit power among all equally-spaced constellations. Draw and label the signal space diagram for this new ternary scheme.

e) [7 points] Given your set of 3 waveforms from part (d), specify the signal space coordinates for a new set of 3 equally-likely waveforms that has the same average probability of error in AWGN channels, but for which the average transmit power is minimum.

a) Each of the signaling waveforms is seen to be a linear combination of $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$, which are orthogonal over the range $0 \le t \le T_b$. Further, we have

$$\int_0^{T_b} \sin^2(2\pi f_c t)\; dt = \frac{T_b}{2}$$

and

$$\int_0^{T_b} \cos^2(2\pi f_c t)\; dt = \frac{T_b}{2}.$$

Thus, we may choose the following orthonormal basis for the signaling waveforms:

$$\begin{aligned} \phi_1(t) &= \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t) \\ \phi_2(t) &= \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t). \end{aligned}$$

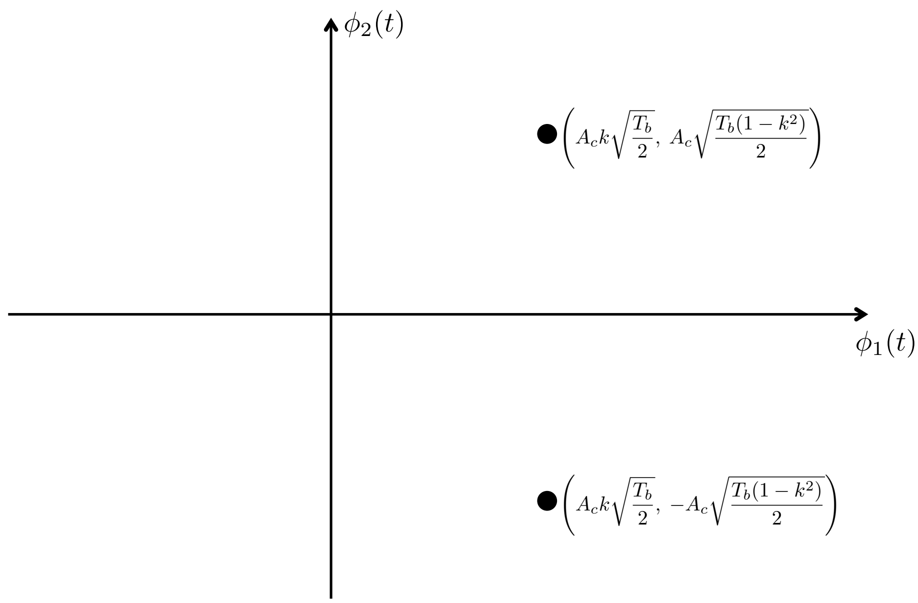
With this, we may write

$$s_1(t) = A_c k\sqrt{\frac{T_b}{2}}\;\phi_1(t) + A_c\sqrt{\frac{T_b(1-k^2)}{2}}\;\phi_2(t),$$

and

$$s_2(t) = A_c k\sqrt{\frac{T_b}{2}}\;\phi_1(t) - A_c\sqrt{\frac{T_b(1-k^2)}{2}}\;\phi_2(t).$$

The corresponding signal space diagram is as shown below.



Note: The subscripts '1' and '2' on the basis functions can, of course, be interchanged. The signal space diagram will change accordingly.

b) Since the signal space component in the $\phi_1(t)$ direction is the same for both signaling waveforms, the optimal detector will depend only on the $\phi_2(t)$ component. In particular, for an observed signal $y(t) = s_i(t)+w(t)$, $i \in \{1,2\}$, the optimal detector will compute the statistic

$$Y = \int_0^{T_b} \phi_2(t)y(t)\ dt,$$

deciding $s_1(t)$ when $Y > 0$ and $s_2(t)$ when $Y < 0$. (For $Y = 0$, either hypothesis may be declared.)

Now, when $s_1(t)$ is sent, Y is conditionally $\mathcal{N}\left(A_c\sqrt{T_b(1-k^2)/2},\ N_0/2\right)$ distributed, and when $s_2(t)$ is sent, Y is conditionally a $\mathcal{N}\left(-A_c\sqrt{T_b(1-k^2)/2},\ N_0/2\right)$ random variable. Thus, we have

$$\begin{aligned} P_e &= \Pr\left(Y < 0 \mid s_1(t) \text{ was sent}\right) \cdot \Pr\left(s_1(t) \text{ was sent}\right) + \Pr\left(Y \geq 0 \mid s_2(t) \text{ was sent}\right) \cdot \Pr\left(s_2(t) \text{ was sent}\right) \\ &= (1/2)Q\left(A_c\sqrt{T_b(1-k^2)/N_0}\right) + (1/2)Q\left(A_c\sqrt{T_b(1-k^2)/N_0}\right) \\ &= Q\left(\sqrt{2E_b(1-k^2)/N_0}\right). \end{aligned}$$

c) The transmitted power (over one symbol period) associated with the carrier component of either waveform is easily shown to be $A_c^2k^2T_b/2$; further, the total transmitted power for either $s_1(t)$ or $s_2(t)$ (over one symbol period) is easily shown to be $A_c^2T_b/2$. Thus, we should choose $k^2 = 0.1$ to satisfy the power conditions specified in the problem.

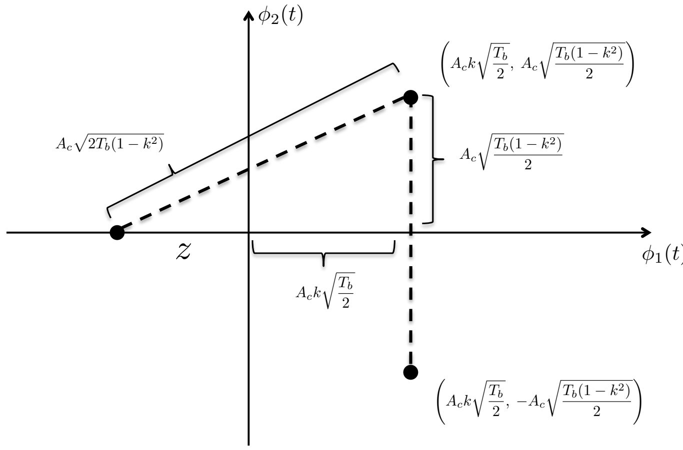
Now, we want to bound the average probability of error to be less than 10^{-4}, which happens here when $Q\left(\sqrt{2E_b(0.9)/N_0}\right) \leq 10^{-4}$. Using the Q-function bound provided, we identify that a sufficient condition to ensure the error probability bound is satisfied is

$$\frac{1}{2}\exp\left(-(0.9)\frac{E_b}{N_0}\right) \leq 10^{-4},$$

since $Q\left(\sqrt{2E_b(0.9)/N_0}\right) \leq \frac{1}{2}\exp\left(-(0.9)\frac{E_b}{N_0}\right)$. This sufficient condition holds provided that

$$\frac{E_b}{N_0} \geq 9.46.$$

d) There are only two choices for the third signaling waveform that satisfy the 'equally spaced' criteria. The one with smaller average transmit power is obtained by solving for 'z' in the diagram below:



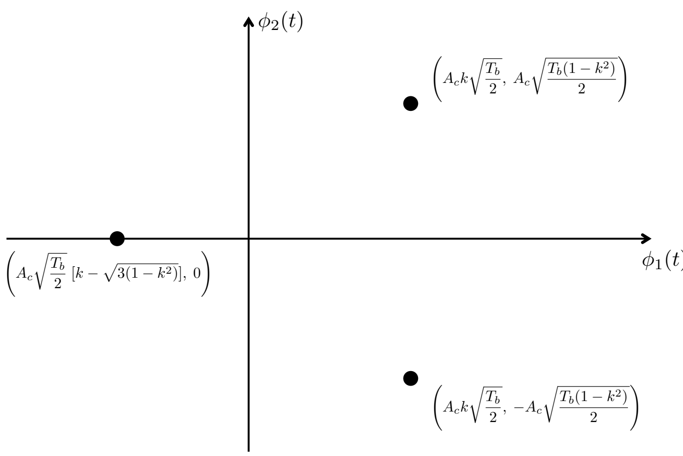
The diagram (with the Pythagorean theorem) implies

$$\left(z + A_c k\sqrt{T_b/2}\right)^2 + \left(A_c\sqrt{T_b(1-k^2)/2}\right)^2 = \left(A_c\sqrt{2T_b(1-k^2)}\right)^2,$$

or, upon simplifying, $z + A_c k\sqrt{T_b/2} = A_c\sqrt{3T_b(1-k^2)/2}$, which yields $z = A_c\sqrt{T_b/2}[\sqrt{3(1-k^2)} - k]$. The signal space coordinates are thus $(-z, 0)$, or $\left(A_c\sqrt{T_b/2}\ [k - \sqrt{3(1-k^2)}],\ 0\right)$, and the corresponding waveform is

$$s_3(t) = A_c[k - \sqrt{3(1-k^2)}]\sin(2\pi f_c t).$$

The new signal space diagram is



Note: The other, higher average power, choice for the third waveform is easily shown to have signal space coordinates

$$(A_c\sqrt{T_b/2}\ [k + \sqrt{3(1-k^2)}],\ 0).$$

e) We can invoke the "invariance" principle, which states that in AWGN channels the probability of error is translation-invariant. So, we can minimize the average transmit power of the constellation from part (d) by subtracting the centroid of the constellation from each of the points.

The centroid of the three points in the constellation is

$$\left(A_c\sqrt{\frac{T_b}{2}}\left[k - \sqrt{\frac{1-k^2}{3}}\right],\ 0 \right).$$

Thus, the new (minimum average transmit power) constellation points are obtained by subtracting this centroid from each of the constellation points, giving

$$\left(A_c\sqrt{\frac{T_b(1-k^2)}{6}},\ A_c\sqrt{\frac{T_b(1-k^2)}{2}} \right),$$
$$\left(A_c\sqrt{\frac{T_b(1-k^2)}{6}},\ -A_c\sqrt{\frac{T_b(1-k^2)}{2}} \right),$$
$$\left(A_c\sqrt{\frac{T_b(1-k^2)}{2}}\left[\sqrt{1/3} - \sqrt{3}\right],\ 0 \right).$$

2) Suppose that you aim to transmit the signal

$$s(t) = 8\sin(3\pi t + \pi/2)$$

using 3-bit PCM.

a) [5 points] What is the Nyquist sampling rate for this signal?

b) [5 points] Suppose the sampling period is $T_s = 1/6$ seconds. Find the PCM waveform for the first 1/2 second.

c) [5 points] For $T_s = 1/6$ seconds, find the mean-square quantization error over the first 1/2 second.

a) The highest frequency component present in the signal has frequency $f_{\max} = 1.5$Hz; thus, the Nyquist sampling rate is $2f_{\max} = 3$ samples/second.

b) There will be four symbols in the first 1/2 second, corresponding to the samples of the original function $s(t) = 8\sin(3\pi t + \pi/2) = 8\cos(3\pi t)$ at times nT_s for $n = 0, 1, 2, 3$. We have

$$\begin{aligned} s(0 \cdot T_s) &= 8\cos(0 \cdot \pi/2) = 8 \\ s(1 \cdot T_s) &= 8\cos(1 \cdot \pi/2) = 0 \\ s(2 \cdot T_s) &= 8\cos(2 \cdot \pi/2) = -8 \\ s(3 \cdot T_s) &= 8\cos(3 \cdot \pi/2) = 0. \end{aligned}$$

Further, since we are using 3 bit PCM, we quantize to 8 levels. The spacing between levels is given by

$$\Delta = \frac{\max_t s(t) - \min_t s(t)}{8} = \frac{16}{8} = 2, \tag{1}$$

so the quantization levels here are given by $\{\pm1, \pm3, \pm5, \pm7\}$.

Thus, the waveform for the first 1/2 second is given by

$$\begin{aligned} s[0] &= 7 \\ s[1] &= 1(\text{ or } -1) \\ s[2] &= -7 \\ s[3] &= 1(\text{ or } -1). \end{aligned}$$

c) We have

$$\begin{aligned} \text{MSE} &= \frac{1}{4}\left[(8-7)^2 + (0-1)^2 + (-8-(-7))^2 + (0-1)^2\right] \\ &= \frac{1}{4}[4] = 1. \end{aligned}$$